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LIMIT STRENGTH OF LOCALLY LOADED SPHERICAL DOMES: RADIAL LOAD AT A CIRCULAR CUT-OUT

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Abstract—The strength of a thin spherical dome loaded radially along the edge of a circular cut-out is determined by using the theorems of plastic limit analysis. Complete solution of the problem comprising of the collapse mode and the associated stress resultant field is obtained. It is shown to be valid for a wide range of values of the shell thickness as well as the sizes of the cut-out and dome. The problem also serves to illustrate how advantage can be taken of the interrelation between the dual static and kinematic problems in constructing a complete solution.

1. INTRODUCTION

Spherical shells, when used as storage vessels or containment chambers, are usually subjected to local loads, such as at the junction where a smaller pipe meets the main shell. A typical example is the junction of a reactor pressure vessel with the ducts connecting it to the stream-rising units. Current interest in such problems is indicated by a recent book (Lukasiewicz, 1979) fully devoted to the elastic analysis of locally loaded plates and shells. There is a vast amount of literature on the elastic analysis of spherical shells under local loads (Reissner, 1946; Flugge and Conrad, 1956; Bijlaard, 1957; Bailey and Hicks, 1960; Leckie, 1961; Koiter, 1963), as well as some papers comparing the theoretical results with the experimental data (Tooth, 1960).

Plastic analysis of spherical shells has been tackled by much fewer researchers. Onat and Prager (1954) performed the limit analysis of the axisymmetric case of a built-in spherical cap subjected to uniformly distributed pressure. Limit pressure for a spherical cap with a cut-out has been obtained by Hodge and co-workers (1963). Dinno and Gill (1965) have treated the problem of obtaining limit pressures for cylinder–sphere junctions, considering the case of a spherical vessel with a cylindrical branch as well as a cylindrical vessel with hemispherical ends. However, all the applications of limit analysis to spherical shell problems quoted above, deal only with non-local loadings such as uniform pressure. As an example of an axisymmetric case of a shell subjected to local loads, we consider the problem described in Section 2.

In the literature on shell limit analysis, in general, very few lower bound solutions have been obtained owing to the difficulties of solving the equilibrium equations and ascertaining the satisfaction of yield condition everywhere in the shell domain. Complete solutions are even more scarce because of the additional demand that the static and the kinematic fields be linked by the flow rule. Of particular interest in obtaining the complete solution to the problem posed below is the simple manner in which advantage can be taken of the interaction between the dual static and kinematic problems: static boundary conditions on two of the stress resultants indicate the nature of the kinematic fields in the plastically deforming parts, which, in turn, give hints regarding the remaining stress resultants. A novel feature of the static solution is the interpretation of the equilibrium equations that some of the stress resultants can be regarded as "generating functions" for the other variables of equilibrium fields, and that prudent choice of these functions aided by kinematics leads to good lower bounds to the limit load. For the problem considered, it is also demonstrated that in the limit state, one of these generating functions is discontinuous. The need for discontinuous statically admissible stress resultant fields occurs very frequently in the limit

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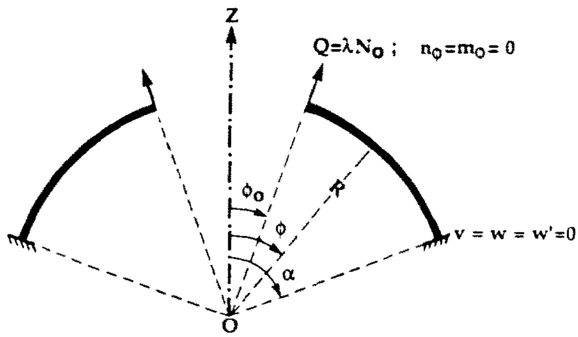

Fig. 1. Cross-section of a clamped spherical dome with a circular cut-out. The applied radial load at the cut-out is indicated by arrows. The stress boundary conditions at the cut-out and the velocity boundary conditions at the clamped edge are as shown.

analysis of shells and a general treatment of such solutions for cylindrical shells has been reported (Srinivasan, 1984).

2. THE PROBLEM

A clamped spherical dome is considered with a circular cut-out at the top (Fig. 1) and subjected to a radial load of constant magnitude Q per unit length on the circumference of the cut-out. The radius of the shell is R and the thickness is T. The angle ϕ_0, a measure of the size of the cut-out, and the dome angle α are both measured from the axis of symmetry OZ as shown.

The dimensionless geometric parameters of the problem and their ranges of values considered are

ε:	non-dimensional thickness $= T/4R$,	$0 \leqslant \varepsilon \leqslant \min(\phi_0, 0.025)$
ϕ_0:	angular size of the cut-out,	$0 \leqslant \phi_0 < \pi/2$
α:	dome angle,	$\phi_0 < \alpha \leqslant \pi/2$.

The validity of the thin shell theory imposes the limits on the values of ε while in the case of ϕ_0 and α, we restrict ourselves to the stated limits.

The material of the shell is assumed to be isotropic and elastic–perfectly plastic with high enough modulus of elasticity that the geometry changes due to elastic deformations are negligible. We wish to determine the limit load Q^* together with the associated velocity and generalized stress fields that satisfy all the governing equations as functions of the parameters that define the geometry of the structure and the yield strength of the constituent material. Theorems of limit analysis are the main tools used.

The geometry of the shell and the type of loading considered possess OZ (Fig. 1) as the axis of symmetry. The only independent variable of the problem is chosen to be ϕ, the angle between the local normal to the shell surface at the point of interest and the axis OZ. It is enough to consider the range $\phi_0 \leqslant \phi \leqslant \alpha$ and all the dependent variables are defined over this interval.

3. STATICS

At a generic point on the shell mid-surface, let $(\mathbf{i}_\phi, \mathbf{i}_\theta, \mathbf{i}_r)$ denote the right-handed triplet of unit vectors that are in the longitudinal, azimuthal and radial directions, respectively. The corresponding stress resultants are nondimensionalized by dividing the actual membrane forces and the transverse shears by $N_0 = \sigma_0 T$ (where σ_0 is the uniaxial yield stress of the shell material) and the bending moments by $M_0 = \sigma_0 T^2/4$. Because of the symmetry present in the geometry of the loaded structure, the non-zero components of stress resultants per unit length of the shell mid-surface are the membrane forces (n_ϕ, n_θ), the bending moments (m_ϕ, m_θ) and the transverse shear, q_ϕ. Since N_0 is the largest membrane force and

M_0, the largest bending moment that a unit length of the shell can carry, the four dependent variables, n_ϕ, n_θ, m_ϕ and m_θ are less than or equal to unity in absolute value.

The dimensionless stress resultants satisfy the equations of equilibrium (see, e.g. Flugge (1960))

$$(q_\phi \sin\phi)' + (n_\theta + n_\phi)\sin\phi = 0 \tag{1a}$$

$$(n_\phi \sin\phi)' - n_\theta \cos\phi - q_\phi \sin\phi = 0 \tag{1b}$$

$$\varepsilon\{(m_\phi \sin\phi)' - m_\theta \cos\phi\} - q_\phi \sin\phi = 0 \tag{1c}$$

where a prime denotes the derivative with respect to ϕ. It is to be noted that eqns (1) were derived under the implicit assumption that all the stress resultants are continuous and differentiable. However, a consideration of discontinuous fields reveals that for the shell to be in equilibrium, the stress resultants n_ϕ, m_ϕ and q_ϕ must be continuous whereas n_θ and m_θ can be discontinuous.

The stress boundary conditions of the problem are

$$n_\phi = m_\phi = 0 \quad \text{on} \quad \phi = \phi_0. \tag{2}$$

Defining a dimensionless load parameter $\lambda = Q/N_0$, we find that the load carried by the shell is related to the stress resultants by

$$\lambda = q_\phi \quad \text{on} \quad \phi = \phi_0. \tag{3}$$

In the usual formulation of shell theory, the transverse shear strain rates are demanded to be zero to the order of approximation considered, resulting in the transverse shear stress resultant q_ϕ playing the role of a "reaction", and hence, the yield condition does not depend on it. Onat and Prager (1954) have obtained such a yield criterion for shells of revolution composed of a material obeying the Tresca yield condition. In general, the yield surface so generated is quite complex to work with. One tractable procedure is to replace it with a simpler surface. The following approximation is chosen that corresponds to a four-dimensional cube of "size" two

$$|n_\phi| \leqslant 1, \quad |n_\theta| \leqslant 1, \quad |m_\phi| \leqslant 1, \quad |m_\theta| \leqslant 1. \tag{4}$$

It can be noted that this approximation is really not suitable for an isotropic shell. However, it captures the essence of the magnitude constraint imposed by yield in a simple manner that is advantageous in energy dissipation calculations and in the establishment of safe equilibrium fields. Moreover, the bounds on the limit load λ^* obtained with yield condition (4) can be used to obtain bounds with any other yield condition by employing the well-known technique of bounding the interior and exterior of the yield surface of interest with the cube (see, e.g. Hodge (1963)).

4. KINEMATICS

In order to construct the dual problem, we let (v, w) denote the dimensionless components of the velocity of the mid-surface of the shell in the local coordinate frame $(\mathbf{i}_\phi, \mathbf{i}_r)$. Moreover, we let ξ denote the dimensionless rate of rotation of a surface element about the $\mathbf{i}_\theta$ direction. For the present problem, the boundary conditions for these field quantities are

$$v = w = \xi = 0 \quad \text{at} \quad \phi = \alpha. \tag{5}$$

The principle of virtual work for the problem is obtained by multiplying each of the equilibrium equations, eqns (1), by (w, v, ξ), respectively, and integrating the sum of the resulting products over $\phi_0 \leqslant \phi \leqslant \alpha$. Assuming the fields to be sufficiently smooth and satisfying boundary conditions (2), (3) and (5), we obtain

$$\lambda w_0 \sin \phi_0 = \int_{\phi_0}^{\alpha} [n_\phi e_\phi + n_\theta e_\theta + \varepsilon m_\phi k_\phi + \varepsilon m_\theta k_\theta + q_\phi \gamma_\phi] \sin \phi \, d\phi \tag{6}$$

where w_0 is the normal velocity w at $\phi = \phi_0$, $e_\phi = v' + w$ and $e_\theta = v \cot \phi + w$ are the membrane strain rates, $k_\phi = \xi'$ and $k_\theta = \xi \cot \phi$ are the "curvature" rates and $\gamma_\phi = \xi + v - w'$ is the transverse shear strain rate.

For yield condition (4) employed here, the accompanying plastic deformation is controlled by the following flow rule:

$$\begin{aligned} \text{when} \quad |n_\phi| < 1, \quad & e_\phi = 0 \\ n_\phi = 1, \quad & e_\phi \geqslant 0 \\ n_\phi = -1, \quad & e_\phi \leqslant 0 \end{aligned} \tag{7}$$

and similar statements relating the remaining stress resultants n_θ, m_ϕ and m_θ to their corresponding duals, e_θ, k_ϕ and k_θ. The usual approximation for thin shells that the transverse shear strain rate is zero implies

$$\xi = -v + w'. \tag{8}$$

The strain rates obtained using eqn (8) are

$$\begin{aligned} e_\phi &= v' + w \\ e_\theta &= v \cot \phi + w \end{aligned} \tag{9a}$$

while the curvature rates become

$$\begin{aligned} k_\phi &= (-v + w')' \\ k_\theta &= (-v + w') \cot \phi. \end{aligned} \tag{9b}$$

Boundary condition (5) can now be restated as

$$v = w = w' = 0 \quad \text{at} \quad \phi = \alpha. \tag{10}$$

5. THE COMPLETE SOLUTION

In the limit analysis of a rigid-perfectly plastic structure, a complete solution is comprised of a statically admissible stress field and a kinematically admissible velocity field such that the strain rates generated are related to the stress field under the limit load by the flow rule at every point in the structure. It is usually difficult to obtain a complete solution, and so one tries to estimate the limit load by obtaining upper and lower bounds to the limit load which are close to each other by using the theorems of limit analysis. Equation (3) gives λ_l, a lower bound to the limit load λ^*, when a statically admissible field satisfying the equations of equilibrium (1), boundary conditions (2), and yield condition (4) is chosen. The upper bound λ_u is obtained by choosing a kinematically admissible function that satisfies velocity boundary conditions (10), and is given by (refer to eqns (6) and (7))

$$\lambda_u = \frac{1}{w_0 \sin \phi_0} \int_{\phi_0}^{\alpha} [|e_\phi| + |e_\theta| + \varepsilon |k_\phi| + \varepsilon |k_\theta|] \sin \phi \, d\phi. \tag{11}$$

The maximization of λ_l or the minimization of λ_u enables λ^* to be determined. However, such an optimization procedure has to be performed over the sets of static or kinematic fields that may exhibit certain discontinuities.

The present aim is to obtain a complete solution to the problem posed. Taking hints from the stress boundary conditions, a kinematically admissible velocity field is chosen. Use of the upper bound theorem provides a pair of coupled transcendental equations, to be solved for an upper bound λ_u and the extent of the plastically deforming region. A statically admissible stress field is then determined which is related to the previously chosen velocity field by the flow rule. It is shown that the determination of the lower bound λ_l leads to the same transcendental equations established before, thus giving the result $\lambda_l = \lambda_u = \lambda^*$ and that we have a complete solution.

The coupled transcendental equations were solved by using elementary numerical methods with the choices of values for ε and ϕ_0 sampling the entire region of validity of the solution in the parameter space. It is further shown that the presence of the small parameter ε enables us to develop an approximate analytical solution. We thus obtain explicitly the dependence of the limit load and the extent of the plastically deforming region on the geometric parameters of the problem.

5.1. *Kinematic solution*

In order to choose a velocity field that is part of a complete solution, a few observations are first made about the stress resultant field. As indicated in Section 3, equilibrium demands that n_ϕ and m_ϕ be continuous everywhere. Since at $\phi = \phi_0$, we have the boundary conditions (2) that $n_\phi = m_\phi = 0$, there must be a region $\phi_0 \leqslant \phi < \phi^*$ within which $|n_\phi| < 1$ and $|m_\phi| < 1$. Flow rule (7) then imposes

$$e_\phi = k_\phi = 0 \quad \text{for} \quad \phi_0 \leqslant \phi < \phi^*. \tag{12}$$

Here ϕ^* is as yet unknown and is assumed to be less than α. Using eqns (9) and solving eqn (12) for v and w which satisfy an additional demand that at $\phi = \phi^*$, $v = w = 0$, we get

$$\left.\begin{aligned} v &= w_0\left\{\frac{1-\cos(\phi^*-\phi)}{\sin(\phi^*-\phi_0)}\right\} \\ w &= w_0\left\{\frac{\sin(\phi^*-\phi)}{\sin(\phi^*-\phi_0)}\right\}. \end{aligned}\right\} \quad \text{for} \quad \phi_0 \leqslant \phi \leqslant \phi^* \tag{13}$$

We choose

$$v = w = 0 \quad \text{for} \quad \phi^* \leqslant \phi \leqslant \alpha. \tag{14}$$

The velocities defined by eqns (13) and (14) constitute a kinematically admissible field with continuous v and w, as shown in Fig. 2. But w' is discontinuous at $\phi = \phi^*$ indicating the presence of a bending hinge.

The choice of eqn (14) implies that the shell remains rigid in the region $\phi^* < \phi \leqslant \alpha$.

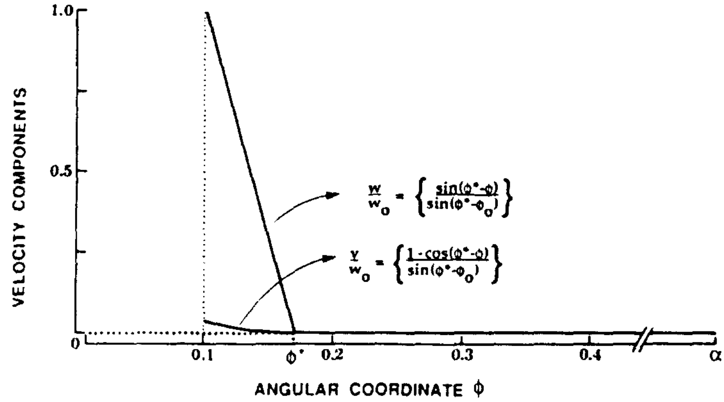

Fig. 2. Velocity field at the limit state (for the case of $\varepsilon = 0.005$ and $\phi_0 = 0.1$). The location of the bending hinge at the limit state is indicated by ϕ^*. The solution is valid for arbitrary values of α, between ϕ^* and $\pi/2$.

In the deforming parts of the shell, the non-zero generalized strain rates as calculated from eqns (9) are

$$e_\theta = w_0 \left\{ \frac{\sin(\phi^* - \phi) + (1 - \cos(\phi^* - \phi)) \cot \phi}{\sin(\phi^* - \phi_0)} \right\} \quad \text{for} \quad \phi_0 \leqslant \phi < \phi^*$$

$$k_\theta = \frac{-w_0 \cot \phi}{\sin(\phi^* - \phi_0)} \quad \text{for} \quad \phi_0 \leqslant \phi < \phi^*$$

$$k_\phi = \frac{w_0 \cos(\phi^* - \phi)}{\sin(\phi^* - \phi_0)} \delta(\phi - \phi^*) \tag{15}$$

where $\delta(\phi - \phi^*)$ is the Dirac delta function.

The upper bound λ_u can now be calculated from eqn (11)

$$\lambda_u \sin \phi_0 \sin(\phi^* - \phi_0) = \varepsilon(2 \sin \phi^* - \sin \phi_0) + \sin \phi^* - \sin \phi_0 - (\phi^* - \phi_0) \cos \phi^*. \tag{16}$$

In the above expression, ϕ^* is unknown. Since the aim is to obtain as close an upper bound to λ^* as possible, ϕ^* is chosen such that λ_u is minimized. Imposing the minimization condition, $d\lambda_u/d\phi^* = 0$ in eqn (16), we get

$$\lambda_u \sin \phi_0 \cos(\phi^* - \phi_0) = 2\varepsilon \cos \phi^* + (\phi^* - \phi_0) \sin \phi^*. \tag{17}$$

The solution of the coupled transcendental equations, eqns (16) and (17), gives the lowest λ_u possible under the chosen velocity field, and the extent of the plastically deforming region as measured by ϕ^*. However, before we try to solve these equations, we establish a statically admissible field related to the velocity field by flow rule (7).

5.2. *Static solution*

Consider the equilibrium equations, eqns (1). For convenience in further arguments, these can be rewritten as

$$q_\phi \sin\phi \cos\phi = \lambda_l \sin\phi_0 \cos\phi_0 - n_\phi \sin^2\phi \tag{18a}$$

$$(n_\phi \sin\phi)' - \frac{1}{\cos\phi}\{\lambda_l \sin\phi_0 \cos\phi_0 - n_\phi \sin^2\phi\} = n_\theta \cos\phi \tag{18b}$$

$$(m_\phi \sin\phi)' - \frac{1}{\varepsilon\cos\phi}\{\lambda_l \sin\phi_0 \cos\phi_0 - n_\phi \sin^2\phi\} = m_\theta \cos\phi. \tag{18c}$$

Here the variable λ is replaced by λ_l to indicate that we are in the process of calculating a lower bound to λ^*. Equation (18a) is obtained by eliminating n_θ from eqns (1a) and (1b), integrating the resulting expressions and substituting conditions (2) and (3). Use of eqn (18a) in eqns (1b) and (1c) results in eqns (18b) and (18c).

According to the lower bound theorem, each statically admissible field satisfying eqns (18), (4) and (2) provides a λ_l as given by eqn (3). The form of eqns (18b) and (18c) suggests the following viewpoint: the variables n_θ and m_θ can be viewed as "generating functions" which can be chosen in the interval $[\phi_0, \alpha]$ such that their magnitudes are less than unity (note that both of them can be discontinuous). Each such choice, in view of eqns (2), (18b) and (18c) provides n_ϕ and m_ϕ fields in the interval $[\phi_0, \alpha]$ as functions of λ_l. By equating the higher value of the maxima of these two fields to unity, we can ensure the satisfaction of expressions (4) and obtain a value for λ_l. The aim is to make prudent choices of n_θ and m_θ such that λ_l is maximized. In order to facilitate this optimization process, we take the following hints from kinematics.

Assuming that the velocity field chosen is part of a complete solution, we have the following conditions on the stress resultant field because of flow rule (7):

$$\text{for } \phi_0 \leqslant \phi < \phi^*, \quad e_\theta > 0 \qquad n_\theta = 1$$
$$\text{implies}$$
$$k_\theta < 0 \qquad m_\theta = -1$$
$$\text{at } \phi = \phi^*, \quad k_\phi > 0 \quad \text{implies} \quad m_\phi = 1. \tag{19}$$

As demanded by expressions (19), $n_\theta = 1$ and $m_\theta = -1$ are imposed in eqns (18b) and (18c) which are then integrated to obtain the following expressions valid in the region $\phi_0 \leqslant \phi \leqslant \phi^*$:

$$n_\phi = (\phi - \phi_0)\cot\phi + \lambda_l \sin\phi_0 \cos\phi_0 (1 - \tan\phi_0 \cot\phi) \tag{20a}$$

$$\varepsilon m_\phi = n_\phi - (1+\varepsilon)\left(1 - \frac{\sin\phi_0}{\sin\phi}\right). \tag{20b}$$

For this region, q_ϕ can be calculated using eqn (18a). Again from expressions (19), we have $m_\phi = 1$ at $\phi = \phi^*$. Imposing this condition in eqn (20b), we get

$$\lambda_l \sin\phi_0 \sin(\phi^* - \phi_0) = \varepsilon(2\sin\phi^* - \sin\phi_0) + \sin\phi^* - \sin\phi_0 - (\phi^* - \phi_0)\cos\phi^*. \tag{21}$$

The restrictions imposed by equilibrium and yield conditions on m'_ϕ at $\phi = \phi^*$ are now investigated. It has already been noted that m_ϕ must be continuous in order to satisfy equilibrium at any ϕ. Let m'_ϕ and m_θ be discontinuous at $\phi = \phi^*$. From eqn (18c), we have

$$[m'_\phi]\sin\phi^* = [m_\theta]\cos\phi^* \tag{22}$$

where the square brackets indicate the difference between the right and the left limits at $\phi = \phi^*$ in the values of the respective variables. The choice $m_\theta = -1$ for $\phi_0 \leqslant \phi < \phi^*$ implies that in order not to violate yield, we must have $2 \geqslant [m_\theta] \geqslant 0$. Since $\phi_0 \leqslant \phi^* \leqslant \pi/2$,

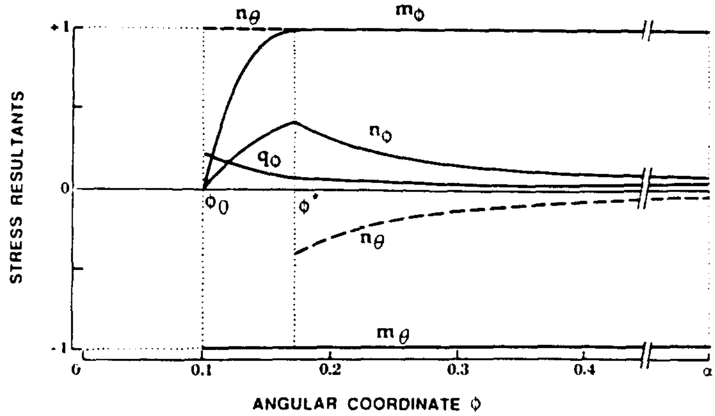

Fig. 3. Stress resultant field at the limit state (for the case of $\varepsilon = 0.005$ and $\phi_0 = 0.1$). Note the discontinuity of n_θ and the slope-discontinuity of n_ϕ at $\phi = \phi^*$. The solution is valid for arbitrary values of α, between ϕ^* and $\pi/2$.

eqn (22) then imposes the condition that $[m'_\phi]$ cannot be negative. It is known from eqn (20b) that m'_ϕ can only be greater than or equal to zero as ϕ approaches ϕ^* from the left, and since $m_\phi = 1$ at $\phi = \phi^*$, m'_ϕ should be non-positive on the right-hand side of ϕ^* for the satisfaction of the yield condition. Thus, the only way all these restrictions on m'_ϕ can be satisfied is by demanding that m'_ϕ be continuous across $\phi = \phi^*$ with its value being equal to zero. We then obtain

$$\lambda_1 \sin \phi_0 \cos (\phi^* - \phi_0) = 2\varepsilon \cos \phi^* + (\phi^* - \phi_0) \sin \phi^*. \tag{23}$$

Equations (21) and (23) can now be solved for λ_1 and ϕ^* in terms of ε and ϕ_0. Comparing these static equations with the kinematic ones, eqns (16) and (17), we see that both pairs are identical, and hence

$$\lambda_1 = \lambda_u = \lambda^*. \tag{24}$$

Thus, we can replace λ_u in eqns (16) and (17), and λ_1 in eqns (21) and (23) by λ^*. As yet, the stress resultant field is not specified in the region $\phi^* < \phi \leqslant \alpha$ which we propose to do next.

Imposing $m'_\phi = 0$ at $\phi = \phi^*$ in eqn (1c), we have

$$q_\phi = 2\varepsilon \cot \phi^* \quad \text{at} \quad \phi = \phi^*. \tag{25}$$

For $\phi^* \leqslant \phi \leqslant \alpha$, we choose $m_\phi = 1$ and $m_\theta = -1$. From eqns (1c), (18a) and (1a), we have

$$q_\phi = 2\varepsilon \cot \phi \tag{26a}$$

$$n_\phi = \frac{\lambda_1 \sin \phi_0 \cos \phi_0}{\sin^2 \phi} - 2\varepsilon \cot^2 \phi \quad \text{for } \phi^* \leqslant \phi \leqslant \alpha \tag{26b}$$

$$n_\theta = 2\varepsilon - n_\phi. \tag{26c}$$

It is easily seen that m_ϕ and q_ϕ are continuous at $\phi = \phi^*$ and the continuity of n_ϕ given by eqns (20a) and (26b) can be established by using eqn (23). The stress resultant fields are as shown in Fig. 3. It can be readily verified that m_ϕ and m_θ do not violate yield conditions (4) anywhere. Equations (20a) (monotonically increasing n_ϕ) and (26b) (monotonically decreasing n_ϕ) imply that n_ϕ attains its maximum at $\phi = \phi^*$. Its value at this location can be obtained from eqn (20b) by using the fact that $m_\phi = 1$ at $\phi = \phi^*$. It is then easy to show

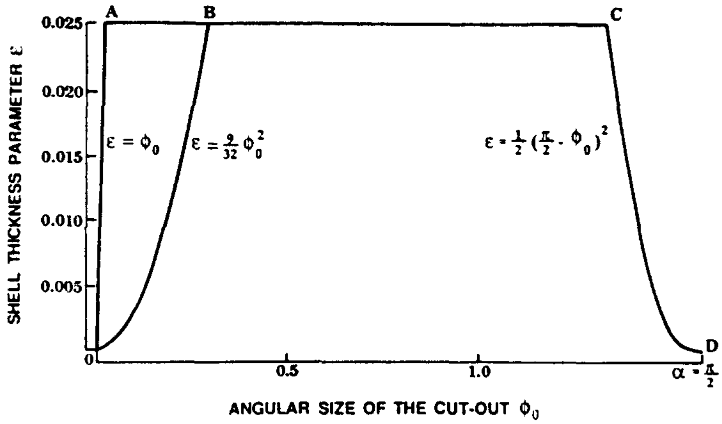

Fig. 4. Validity region of the solution in the parameter space when $\alpha = \pi/2$. The internal border OB separates the two regions defined by the approximate analytical solution (31).

that for the parameter values of interest (region OABCD of Fig. 4), n_ϕ does not violate yield. The function n_θ being 1 for $\phi_0 \leqslant \phi < \phi^*$ and differing from n_ϕ only by 2ε (note that $\varepsilon \leqslant 0.025$) in magnitude in the region $\phi^* < \phi \leqslant \alpha$ also satisfies the yield condition. Thus, a statically admissible field has been established which together with the velocity field defined by eqns (13) and (14) provides a complete solution to the problem.

5.3. *Approximations*

We observe that λ^* and ϕ^* are independent of the dome angle α and the complete solution is valid if ϕ^* is less than α. By knowing ε and ϕ_0, we can solve eqns (21) and (23) numerically for λ^* and ϕ^*. For convenience in following such a procedure, we eliminate λ^* from eqns (21) and (23) to get

$$\varepsilon g(\Delta, \phi_0) + h(\Delta, \phi_0) = 0 \tag{27}$$

where

$$g(\Delta, \phi_0) = \frac{\sin \phi_0}{\cos \Delta}(\cos \Delta - 2)$$

$$h(\Delta, \phi_0) = \frac{\cos \phi_0}{\cos \Delta}(\Delta - \sin \Delta \cos \Delta) + \sin \phi_0(1 - \cos \Delta)$$

and

$$\Delta = \phi^* - \phi_0.$$

Equation (27) can now be solved for Δ by using an iterative numerical scheme and the resulting value of ϕ^* would enable the calculation of λ^* in eqn (23). However, an analytical solution, even if approximate, is desirable as it gives explicitly the form of dependence of the strength of the structure on the parameters of the problem. The key to an approximate analytical solution in the present case is the observation that ε is a small parameter.

Consider now the limiting case of $\varepsilon = 0$. From eqn (27), we have $h(\Delta, \phi_0) = 0$, which can be written as

$$\tan \phi_0(1 - \cos \Delta) + \left(\frac{\Delta}{\cos \Delta} - \sin \Delta\right) = 0 \tag{28}$$

where ϕ_0 has a given value between 0 and $\pi/2$. Since we only consider cases where $\alpha \leqslant \pi/2$, we must have $0 \leqslant \Delta \leqslant \pi/2$. Thus, $\tan \phi_0$ as well as the terms within both parentheses are

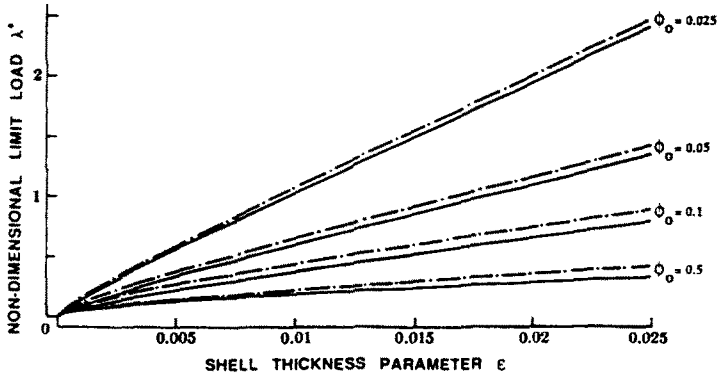

Fig. 5(a). Non-dimensional limit load as a function of the non-dimensional shell thickness for representative values of the angular size of the cut-out. The solid lines are obtained by numerically solving eqn (27), and substituting the results in eqn (23). The dashed lines are given by the approximate analytical solution (32).

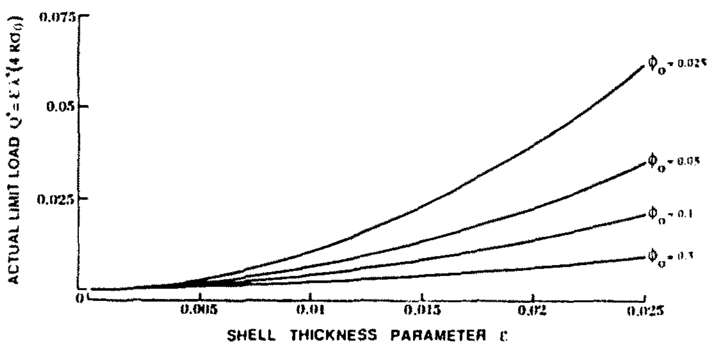

Fig. 5(b). Actual limit load as a function of the non-dimensional shell thickness for representative values of the angular size of the cut-out. The curves are generated by using the numerical solution for λ^* (shown in Fig. 5(a)) in eqn (33). The constant multiplier $4R\sigma_0$ is taken to be unity.

non-negative. Therefore, the only way eqn (28) can be satisfied for a given ϕ_0 is by equating the terms within parentheses to zero. This implies that $\Delta = 0$ when $\varepsilon = 0$.

As ε increases from zero, since eqn (27) is composed of smooth functions, we expect Δ to increase gradually. Because ε has to be less than about 0.025 for thin shell theory to be valid, we adopt the following procedure. We approximate $\sin\Delta$ and $\cos\Delta$ by retaining only a few terms of their respective infinite power series in Δ. We solve the resulting equations with further approximation and then show that the neglection of higher order terms was indeed justified within the domain of interest of the parameter space. We also compare, for certain chosen values of the parameters, the approximate analytical solution with that obtained by applying an iterative procedure to solve eqn (27).

Equation (27) can be approximated as

$$-\varepsilon\sin\phi_0+\tfrac{2}{3}\Delta^3\cos\phi_0+\frac{\Delta^2}{2}\sin\phi_0\simeq 0 \tag{29}$$

noting that $\varepsilon \ll 1$ and defining $c = (4\sqrt{(2\varepsilon)}/3)\cot\phi_0$ and $\delta = \Delta/\sqrt{(2\varepsilon)}$ we have

$$c\delta^3+\delta^2-1=0 \tag{30}$$

where $0 < \phi_0 < \pi/2$ implies $\infty > c > 0$. An approximate solution to eqn (30) is

$$\begin{aligned}\delta &\simeq 1 &&\text{for } 0 < c \leqslant 1\\ &\simeq c^{-1/3} &&\text{for } 1 \leqslant c \leqslant \infty.\end{aligned} \tag{31}$$

Again, approximating $\sin\Delta$ and $\cos\Delta$ in eqn (23), we get

$$\lambda^* \simeq \sqrt{(2\varepsilon)}\{\delta+\sqrt{(2\varepsilon)}\cot\phi_0(1+\delta^2)\} \tag{32}$$

where δ is given by eqns (31). Equations (31) and (32) give explicit expressions for the extent of the plastically deforming region and the limit load, respectively, as functions of ε and ϕ_0.

5.4. *Region of validity*

The region of validity of the solution obtained in the parametric space (ϕ_0, ε) is now established. As shown in Fig. 4, its border OABCD is determined by the following constraints.

(1) The parameter $\varepsilon = T/4R$ is, by definition, positive and for thin shell theory to reasonably approximate the actual three-dimensional solution, $0 \leqslant \varepsilon \leqslant 0.025$.

(2) A general conclusion that can be reached by a study of structural members subjected to highly localized loading is that only if the diameter of the loaded region is at least of the order of the shell thickness would a limit load obtained by using thin shell theory be close to a three-dimensional solution (refer to Anderson and Shield (1966)). Otherwise, the deformation near the load would not be through the thickness. Therefore, we restrict ourselves to the region $\varepsilon \leqslant \phi_0$.

(3) For the solution obtained to be valid, we must have $\phi^* \leqslant \alpha$. This constraint can be rewritten as $\phi_0+\Delta \leqslant \alpha$. Line CD represents this condition for the limiting case of $\alpha = \pi/2$ with the equation $\phi_0+\sqrt{(2\varepsilon)} = \pi/2$.

(4) Line OB is given by the condition $c = 1$, and represents the border between the two regions defined by the approximate analytical solution (31).

Now it only remains to show that eqn (29) is a good approximation to eqn (27). An examination of the values attained by Δ in the region OABC of Fig. 4 shows that it reaches its maximum along BC and is equal to 0.32. For this value, the error involved in taking $\sin\Delta \cong \Delta$ and $\cos\Delta \cong 1$ is less than 5%. For certain chosen values of ϕ_0, the variation of λ^* with respect to ε as given by both the numerical and approximate analytical solutions are shown in Fig. 5(a). We observe that the non-dimensional limit load λ^* increases rapidly when ε is extremely small with $d\lambda^*/d\varepsilon$ being infinite at $\varepsilon = 0$. However, the actual limit load is given by

$$Q^* = \lambda^* N_0 = \varepsilon\lambda^*(4R\sigma_0). \tag{33}$$

The variations of Q^* with ε are shown in Fig. 5(b). The collapse mode corresponding to the limit load is given by velocity fields (13) and (14) resulting in the opening up of a small annular region of angular width Δ (at most, equal to $\sqrt{(2\varepsilon)}$).

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REFERENCES

Anderson, C. A. and Shield, R. T. (1966). On the validity of the plastic theory of structures for collapse under highly localized loading. *J. Appl. Mech.* **33**, 629–636.
Bailey, R. and Hicks, R. (1960). Localized loads applied to a spherical pressure vessel through a cylindrical insert. *J. Mech. Engng Sci.* **2**, 302–311.
Bijlaard, P. P. (1957). Computation of the stresses from local loads in spherical pressure vessels or pressure vessel heads. Welding Research Council Bulletin No. 34.
Dinno, K. S. and Gill, S. S. (1965). The limit analysis of a pressure vessel consisting of the junction of a cylindrical and spherical shell. *Int. J. Mech. Sci.* **7**, 21–42.
Flugge, W. and Conrad, D. A. (1956). Singular solutions in the theory of shallow shells. Stanford Univ., Div. Engng Mech. Report No. 101.
Flugge, W. (1960). *Stresses in Shells*. Springer, Berlin.
Hodge, P. G. (1963). *Limit Analysis of Rotationally Symmetric Plates and Shells*. Prentice-Hall, Englewood Cliffs, New Jersey.
Koiter, W. T. (1963). A spherical shell under point loads at its poles. *Adv. Appl. Mech.* Prager Anniversary Volume, 155–169.

Leckie, F. A. (1961). Localized loads applied to spherical shells. *J. Mech. Engng Sci.* **3**, 111–118.
Lukasiewicz, S. (1979). *Local Loads on Plates and Shells.* PWN-Polish Scientific Publishers, Warszawa.
Srinivasan, M. A. (1984). Strength of shells under local transverse loads. Ph.D. Thesis, Yale University.
Onat, E. T. and Prager, W. (1954). Limit analysis of shells of Revolution I and II. *Konink1. Nederl. Academie Van Wetenschapen, Amsterdam Proceedings*, Series B57, No. 5.
Reissner, E. (1946). Stresses and small displacements of shallow spherical shells, II. *J. Math. Phys.* **25**, 279.
Tooth, A. S. (1960). An experimental investigation of shallow spherical domes subjected to a variety of load actions. *Symp. Nuclear Reactor Containment Buildings and Pressure Vessels*, Glasgow.